LEZIONE 4 (7/09/2017)

CRASH COURSE IN ANALISI MATEMATICA

DISEQUAZIONI IRRAZIONALI

Analizziamo la risoluzione di disequazioni irrazionali (disequazioni algebriche in cui la x compare sotto il simbolo di radice) nei seguenti casi:

(1) $$A(x) > \sqrt{B(x)} \iff \begin{cases} B(x) \geq 0 \\ A(x) > 0 \\ [A(x)]^2 > B(x) \end{cases}$$

(2) $$A(x) < \sqrt{B(x)} \iff \begin{cases} B(x) \geq 0 \\ A(x) < 0 \end{cases} \cup \begin{cases} A(x) \geq 0 \\ [A(x)]^2 < B(x) \end{cases}$$

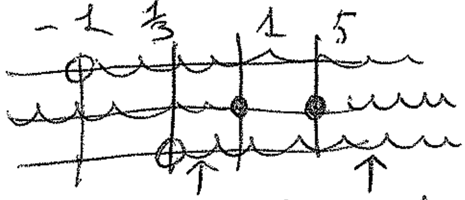
(3) $$A(x) \gtrless \sqrt[3]{B(x)} \iff [A(x)]^3 \gtrless B(x)$$

(4) $$\sqrt{A(x)} \gtrless \sqrt{B(x)} \iff \begin{cases} A(x) \geq 0 \\ B(x) \geq 0 \\ A(x) \gtrless B(x) \end{cases}$$

<u>Esercizio</u> $x+1 > \sqrt{x^2-4x+3}$ (caso(1))

$$\begin{cases} x^2-4x+3 \geq 0 \\ x+1 > 0 \\ (x+1)^2 > x^2-4x+3 \end{cases} \Rightarrow \begin{cases} x \leq 1 \ \vee \ x \geq 3 \\ x > -1 \\ 2x+1 > -4x+3 \end{cases}$$

$$\begin{cases} x \leq 1 \ \vee \ x \geq 3 \\ x > -1 \\ x > \frac{1}{3} \end{cases}$$

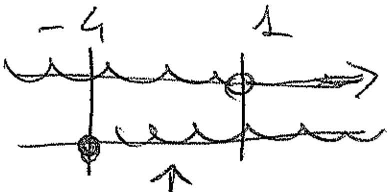


S: $\frac{1}{3} < x \leq 1 \ \vee \ x \geq 5$

Esercizio $x-1 < \sqrt{x+4}$ (caso (2))

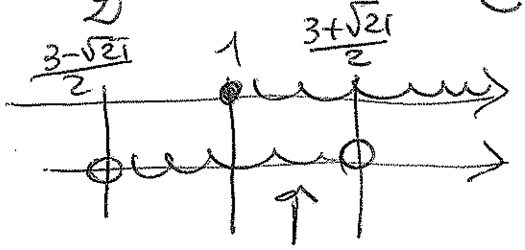
$$\begin{cases} x-1<0 \\ x+4 \geq 0 \end{cases} \cup \begin{cases} x-1 \geq 0 \\ (x-1)^2 < x+4 \end{cases}$$

$$\begin{cases} x<1 \\ x \geq -4 \end{cases} \cup \begin{cases} x \geq 1 \\ x^2-3x-3<0 \end{cases}$$



$S_1 = [-4, 1[$

$$\begin{cases} x \geq 1 \\ \frac{3-\sqrt{21}}{2} < x < \frac{3+\sqrt{21}}{2} \end{cases}$$



$S_2 = [1, \frac{3+\sqrt{21}}{2}[$

$S = S_1 \cup S_2 = [-4; \frac{3+\sqrt{21}}{2}[$

Esercizio $x-3 < \sqrt[3]{x^3+3x^2-x-27}$ (caso (3))

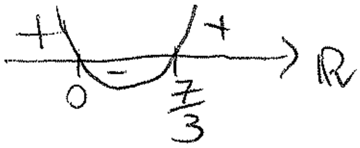
$$(x-3)^3 < x^3+3x^2-x-27$$

$$\cancel{x^3} \cancel{-27} + 27x - 9x^2 < \cancel{x^3} + 3x^2 - x \cancel{-27}$$

$$12x^2 - 28x > 0$$

$$3x^2 - 7x > 0$$

$$x(3x-7) > 0$$

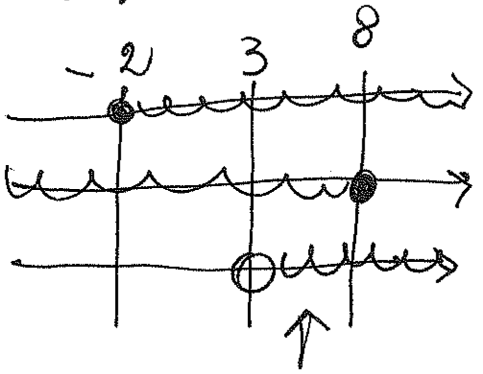


$S: x<0 \vee x > \frac{7}{3}$

Esercizio $\sqrt{x+2} - \sqrt{8-x} > 0$

$$\sqrt{x+2} > \sqrt{8-x}$$

$$\begin{cases} x+2 \geq 0 \\ 8-x \geq 0 \\ x+2 > 8-x \end{cases} \quad \begin{cases} x \geq -2 \\ x \leq 8 \\ 2x > 6 \end{cases} \quad \begin{cases} x \geq -2 \\ x \leq 8 \\ x > 3 \end{cases}$$

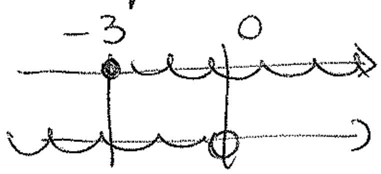


$$S: \quad 3 < x \leq 8$$

Esercizio $\sqrt{x+3} > x \iff \begin{cases} x+3 \geq 0 \\ x < 0 \end{cases} \cup \begin{cases} x \geq 0 \\ x+3 > x^2 \end{cases}$

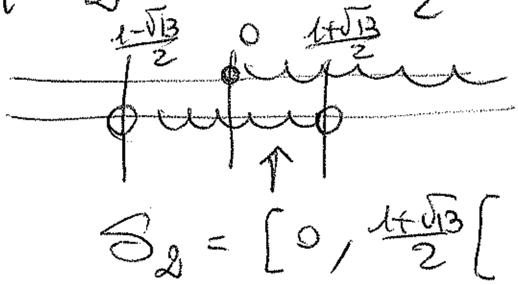
$$\iff \begin{cases} x \geq -3 \\ x < 0 \end{cases} \cup \begin{cases} x \geq 0 \\ x^2 - x - 3 < 0 \end{cases}, \quad x^2 - x - 3 = 0$$

$$x_{1,2} = \frac{1 \pm \sqrt{13}}{2}$$



$$S_1 = [-3, 0[$$

$$\begin{cases} x \geq 0 \\ \frac{1-\sqrt{13}}{2} < x < \frac{1+\sqrt{13}}{2} \end{cases}$$



$$S_2 = \left[0, \frac{1+\sqrt{13}}{2}\right[$$

$$S = S_1 \cup S_2 = \left[-3, \frac{1+\sqrt{13}}{2}\right[$$

Esercizio

$$\underbrace{\sqrt{x-4}}_{\geq 0} > -\underbrace{\sqrt{x+2}}_{\geq 0},$$

$$\begin{cases} x \geq 4 \\ x \geq -2 \end{cases} \Rightarrow \quad x \geq 4$$

Poichè il primo membro, per $x \geq 4$, risulta ≥ 0, mentre il secondo membro è < 0, allora la disuguaglianza assegnata risulta verificata.

$$S = [4, +\infty[$$

Esercizio $\sqrt{\frac{x+1}{x-1}} \leq 4 \iff$

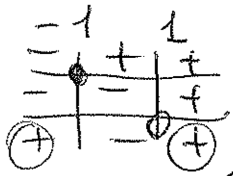
$$\iff \begin{cases} \frac{x+1}{x-1} \geq 0 \\ \frac{x+1}{x-1} \leq 16 \end{cases} \iff \frac{x+1-16x+16}{x-1} \leq 0$$

$$\iff \begin{cases} \frac{x+1}{x-1} \geq 0 & (1) \\ \frac{-15x+17}{x-1} \leq 0 \end{cases} \iff \frac{15x-17}{x-1} \geq 0 \quad (2)$$

(1) $\frac{x+1}{x-1} \geq 0$

$x+1 \geq 0 \Rightarrow x \geq -1$

$x-1 > 0 \Rightarrow x > 1$

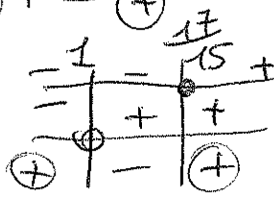


$x \leq -1 \vee x > 1$

(2) $\frac{15x-17}{x-1} \geq 0$

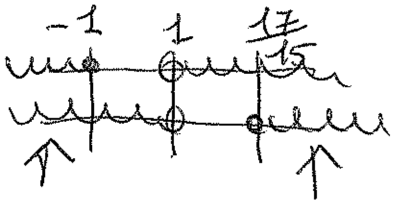
$15x-17 \geq 0 \Rightarrow x \geq \frac{17}{15}$

$x-1 > 0 \Rightarrow x > 1$



$x < 1 \vee x \geq \frac{17}{15}$

$$\begin{cases} x \leq -1 \vee x > 1 \\ x < 1 \vee x \geq \frac{17}{15} \end{cases}$$



$S: x \leq -1 \vee x \geq \frac{17}{15}$

Esercizio $\sqrt{x^2-4} > x$

$$\begin{cases} x^2-4 \geq 0 \\ x < 0 \end{cases} \cup \begin{cases} x \geq 0 \\ \cancel{x^2}-4 > \cancel{x^2} \end{cases}$$

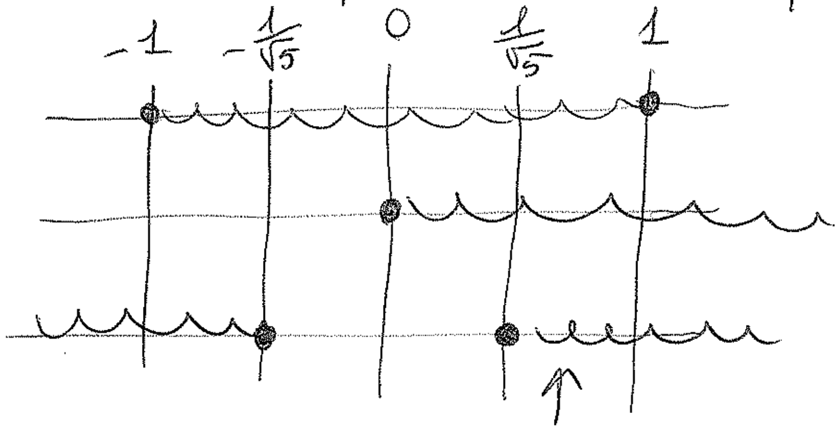
$$\begin{cases} x \leq -2 \vee x \geq 2 \\ x < 0 \end{cases} \cup \begin{cases} x \geq 0 \\ \nexists x \in \mathbb{R} \end{cases}$$

$S_1 : x \leq -2$ $\qquad S_2 = \emptyset$

$S = S_1 \cup S_2 : x \leq -2$

Esercizio $\sqrt{1-x^2} \leq 2x$

$$\begin{cases} 1-x^2 \geq 0 \\ 2x \geq 0 \\ 1-x^2 \leq 4x^2 \end{cases} \Rightarrow \begin{cases} -1 \leq x \leq 1 \\ x \geq 0 \\ 5x^2-1 \geq 0 \end{cases} \Rightarrow \begin{cases} -1 \leq x \leq 1 \\ x \geq 0 \\ x \leq -\frac{1}{\sqrt5} \vee x \geq \frac{1}{\sqrt5} \end{cases}$$



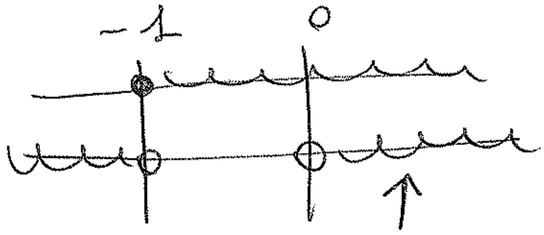
$S: \frac{1}{\sqrt5} \leq x \leq 1$

$S = \left[\frac{1}{\sqrt5} ; 1\right]$

Esercizio:

$$\underbrace{|1+x|}_{\substack{\vee\!\!\mid \\ 0\ \forall x\in\mathbb{R}}} > \sqrt{1+x} \Rightarrow \begin{cases} 1+x \geq 0 \\ (1+x)^2 > 1+x \end{cases} \Rightarrow \begin{cases} x \geq -1 \\ \cancel{1}+x^2+2x > \cancel{1}+x \end{cases}$$

$$\begin{cases} x \geq -1 \\ x^2+x>0 \end{cases} \Rightarrow x(x+1)>0 \Rightarrow \begin{cases} x \geq -1 \\ x<-1 \vee x>0 \end{cases}$$



$S: x>0$

$S =]0, +\infty[$

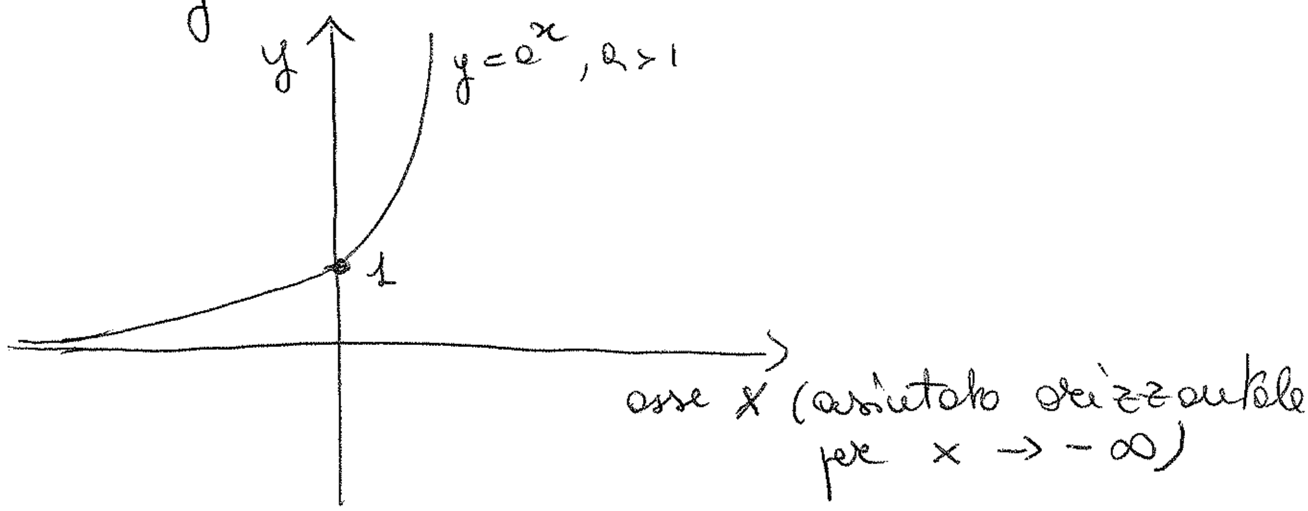
__Funzione esponenziale__

$y=f(x)=a^x$, $a\in\mathbb{R}$, $a>0$, $a\neq 1$

$f: \mathbb{R} \to \mathbb{R}$

$D=\mathbb{R}$, $\mathrm{im}(f)=\mathbb{R}^+ =]0,+\infty[$

Grafico di $y=a^x$ con $a>1$:



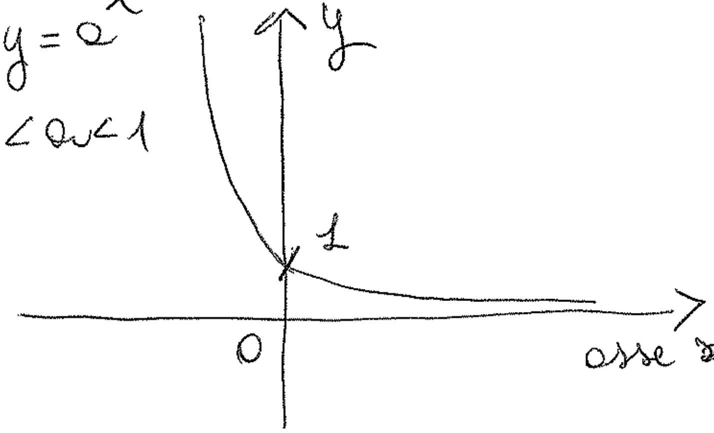
se $a>1$:

$$\lim_{x\to-\infty} a^x = 0^+ \quad ; \quad \lim_{x\to+\infty} a^x = +\infty$$

La funzione $y=a^x$ è crescente in senso stretto se $a>1$:

$$\forall x_1,x_2 \in D=\mathbb{R}:\quad x_1<x_2 \iff a^{x_1}<a^{x_2}$$

Grafico di $y=a^x$ con $0<a<1$:



asse x (asintoto orizzontale per $x\to+\infty$)

$$\lim_{x\to-\infty} f(x) = +\infty \quad ; \quad \lim_{x\to+\infty} f(x) = 0^+$$

La funzione $y=a^x$ è strettamente decrescente se $0<a<1$:

$$\forall x_1,x_2 \in D=\mathbb{R}:\quad x_1<x_2 \iff a^{x_1}>a^{x_2}.$$

N.B. Se $a=1$: $a^x=1^x=1$, $y=1$ è una funzione costante, il cui grafico è una retta orizzontale.

Se $a<0$, ad esempio $a=-3$, $y=a^x=(-3)^x$. L'immagine di $x=\frac{3}{4}$ con tale funzione sarebbe

$$f\left(\tfrac{3}{4}\right)=(-3)^{\frac{3}{4}}=\sqrt[4]{(-3)^3}=\sqrt[4]{-27}\notin\mathbb{R},$$

ecco perché si considera $a>0$.

Proprietà delle potenze

$$a^0 = 1 \qquad (a \neq 0)$$

$$a^1 = a$$

$$a^n \cdot a^m = a^{n+m}$$

$$a^n : a^m = a^{n-m}$$

$$(a \cdot b)^n = a^n \cdot b^n$$

$$(a^n)^m = a^{n \cdot m}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad , \quad b \neq 0$$

$$a^{-1} = \frac{1}{a} \quad , \quad a \neq 0$$

$$a^{-n} = \frac{1}{a^n} \quad , \quad a \neq 0$$

$$a^{\frac{n}{m}} = \sqrt[m]{a^n} \quad , \quad a > 0$$

$$a^{-\frac{n}{m}} = \frac{1}{\sqrt[m]{a^n}} \qquad a > 0$$

Equazioni e disequazioni esponenziali

1) $\boxed{(5^x)^2 = \sqrt{5}} \iff 5^{2x} = 5^{\frac{1}{2}} \Rightarrow 2x = \frac{1}{2}$

$\Rightarrow x = \frac{1}{4}$

2) $\boxed{\left(\frac{3}{4}\right)^{-2x} = \left(-\frac{4}{3}\right)^{-2}} \qquad \left(\frac{4}{3}\right)^{2x} = \left(\frac{4}{3}\right)^{-2} \Rightarrow 2x = -2$

$x = -1$

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3) $8^{|x-2|} = 2\sqrt{2}$

$2^{3|x-2|} = 2 \cdot 2^{\frac{1}{2}} \Rightarrow 2^{3|x-2|} = 2^{1+\frac{1}{2}}$

$\Rightarrow \cancel{3}|x-2| = \frac{\cancel{3}}{2} \Rightarrow |x-2| = \frac{1}{2}$

$\Rightarrow x-2 = \pm\frac{1}{2}$ ↗ $x-2 = \frac{1}{2} \Rightarrow x = \frac{5}{2}$

↘ $x-2 = -\frac{1}{2} \Rightarrow x = \frac{3}{2}$

$S = \{\frac{3}{2}, \frac{5}{2}\}$

4) $e^{1-|x-1|} < \frac{1}{\sqrt{e}}$

$e^{1-|x-1|} < e^{-\frac{1}{2}} \Rightarrow 1-|x-1| < -\frac{1}{2}$

$\Rightarrow |x-1| > 1+\frac{1}{2} \Rightarrow |x-1| > \frac{3}{2}$

$\Rightarrow x-1 < -\frac{3}{2} \vee x-1 > \frac{3}{2}$

S: $\boxed{x < -\frac{1}{2} \vee x > \frac{5}{2}}$

5.4

5) $\boxed{2^{x+1} + 2^{x+2} + 2^{x} \leq 7 \cdot 2^{3-x}}$

$$2 \cdot 2^{x} + 4 \cdot 2^{x} + 2^{x} \leq 7 \cdot 2^{3-x}$$

$$2^{x}\cancel{(2+4+1)} \leq \cancel{7} \cdot 2^{3-x}$$

$$x \leq 3 - x$$

$$2x \leq 3 \Rightarrow x \leq \frac{3}{2}$$

$$S = \left]-\infty ; \frac{3}{2}\right]$$

Exercício

6) $2^{x+1} + 2^{-x} = 3$

$$2 \cdot 2^{x} + \frac{1}{2^{x}} = 3 \Rightarrow 2 \cdot 2^{2x} + 1 = 3 \cdot 2^{x}$$

$$\Rightarrow 2 \cdot 2^{2x} - 3 \cdot 2^{x} + 1 = 0$$

$$2^{x} = t$$

$$2t^{2} - 3t + 1 = 0$$

$$-2t - t$$

$$2t(t-1) - (t-1) = 0 \Rightarrow (t-1)(2t-1) = 0$$

$t = 1 \vee t = \frac{1}{2}$

$2^{x} = 1 = 2^{0}$ $\quad$ $2^{x} = 2^{-1}$

$x = 0$ $\quad$ $x = -1$ $\quad$ $S = \{0, -1\}$

Esercizio

f) $e^{2x} - e^{x+1} + e^{x} - e < 0$

$e^{2x} - e \cdot e^{x} + e^{x} - e < 0$

$e^{2x} + e^{x}(1-e) - e < 0$

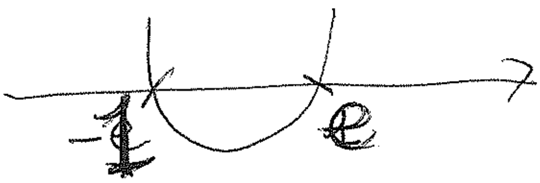
$e^{x} = t$

$t^2 - (-1+e)t - e < 0$

$t_{1,2} = \dfrac{-1+e \pm \sqrt{(-1+e)^2 + 4e}}{2} = \dfrac{-1+e \pm \sqrt{1+e^2-2e+4e}}{2} =$

$= \dfrac{-1+e \pm \sqrt{e^2+1+2e}}{2} = \dfrac{-1+e \pm \sqrt{(e+1)^2}}{2} =$

$= \dfrac{-1+e \pm \overset{|e+1|}{(e+1)}}{2} = \begin{cases} \dfrac{\cancel{-1}+e+e\cancel{+1}}{2} = \dfrac{\cancel{2}e}{\cancel{2}} = e \\[2ex] \dfrac{-1\cancel{+e}\cancel{-e}-1}{2} = -\dfrac{2}{2} = -1 \end{cases}$



$-1 < t < e$

$\Downarrow$

$\underbrace{\overset{<0}{-1} < e^{x}}_{\forall x \in \mathbb{R}} < e \Rightarrow e^{x} < e = e^{1}$

$x < 1$

Esercizio 8)

$$\boxed{\frac{e^{4-x^2}-1}{\sqrt{e}-e^{x-3}} \geq 0}$$

$N \geq 0 \Rightarrow e^{4-x^2}-1 \geq 0$

$e^{4-x^2} \geq 1$

$e^{4-x^2} \geq e^0 \Rightarrow 4-x^2 \geq 0$

$\Rightarrow -2 \leq x \leq 2$

$D > 0 \Rightarrow \sqrt{e}-e^{x-3} > 0 \Rightarrow e^{x-3} < e^{\frac{1}{2}} \Rightarrow x-3 < \frac{1}{2}$

$\Rightarrow x < \frac{7}{2}$

		-2		2		$\frac{7}{2}$	
N	$-$	$\bullet$	$+$	$\bullet$	$-$		$-$
D	$+$		$+$		$+$	$\circ$	$-$
F	$-$		$+$		$-$		$+$

$$\boxed{S: -2 \leq x \leq 2 \vee x > \frac{7}{2}}$$

Disequazione esponenziale riconducibile ad una disequazione di 2° grado

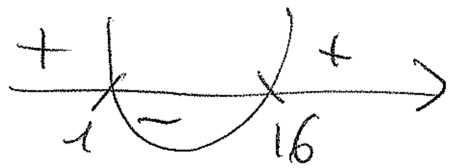
Esercizio 9)

$$\boxed{4^x - 17 \cdot 2^x + 16 > 0}$$

$2^x = t$

$t^2 - 17t + 16 > 0$

$(t-16)(t-1) > 0$



$t < 1 \vee t > 16$

$2^x < 2^0 \vee 2^x > 2^4$

$$\boxed{x < 0 \vee x > 4}$$

10) $\boxed{\dfrac{e^{-3x} - \sqrt{e^{2x-1}}}{e^{2x}} > 0}$

Poichè il denominatore è $e^{2x} > 0 \quad \forall x \in \mathbb{R}$:

$$e^{-3x} - \sqrt{e^{2x-1}} > 0$$

$$e^{-3x} > e^{\frac{2x-1}{2}} \Rightarrow -3x > \frac{2x-1}{2}$$

$$\Rightarrow -6x > 2x - 1 \Rightarrow 8x < 1$$

$$S: \; x < \frac{1}{8}$$

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11) $\dfrac{e^x - 1}{(e^x - e^3)(\sqrt{e} - e^x)} \leq 0$

$N \geq 0 \Rightarrow e^x - 1 \geq 0 \Rightarrow e^x \geq 1 = e^0 \Rightarrow x \geq 0$

D: $e^x - e^3 > 0 \Rightarrow e^x > e^3 \Rightarrow x > 3$

$\sqrt{e} - e^x > 0 \Rightarrow e^{\frac{1}{2}} > e^x \Rightarrow x < \frac{1}{2}$

		0		$\frac{1}{2}$		3	
N	$-$		$+$		$+$		$+$
D_1	$-$		$-$		$-$		$+$
D_2	$+$		$+$		$-$		$-$
F:	$+$		$-$		$+$		$-$

$$S: \; 0 \leq x < \frac{1}{2} \;\vee\; x > 3$$

Sistema di disequazioni esponenziali

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$$\begin{cases} \dfrac{|5^x-1|}{2^x-4} \le 0 & (1) \\ (0,\bar{3})^x > \sqrt{3} & (2) \end{cases} \Rightarrow \begin{cases} x < 2 \\ x < -\frac{1}{2} \end{cases} \Rightarrow S: x < -\frac{1}{2}$$

(1) $|5^x-1| \ge 0 \quad \forall x \in \mathbb{R}$

$|5^x-1| = 0$ per $5^x = 5^0 \Rightarrow x = 0$

deve appartenere alla soluz. visto che non annulla il denominatore

$\frac{0}{-3} \le 0$

$0 \le 0$

vero

$D < 0 \Rightarrow 2^x - 4 < 0$

$2^x < 2^2$

$S: x < 2$

N.B. Nella soluzione è compreso $x = 0$.

(2) $(0,\bar{3})^x > \sqrt{3}$

$\left(\frac{1}{3}\right)^x > 3^{\frac{1}{2}}$

$\left(\frac{1}{3}\right)^x > \left(\frac{1}{3}\right)^{-\frac{1}{2}} \Rightarrow x < -\frac{1}{2}$

$$\begin{cases} x < 2 \\ x < -\frac{1}{2} \end{cases} \Rightarrow x < -\frac{1}{2}$$